INFINITE DERIVATIVE'S SERIES EXPANSION OF INDEFINITE AND DEFINITE INTEGRAL

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Abstract. In this article it is proven the existence of integration of indefinite integrals as infinite derivative's series expansion. $\int f(x)dx = \sum_{i=0}^{\infty}(-1)^i f^{(i)}(x)\frac{x^{i+1}}{(i+1)!} + C$. This also opens a new way to integrate a definite integral.

1. Introduction

Numerical integration is used in science nowadays, e.g. [2] [3], but these methods are not as flexible as symbolic math is.
Symbolic computation of indefinite integrals was developed by Liouville, Risch and Bronstein ([4], [5]) , but Rish algorithm can not solve a lot of cases.
Approximation in integration with Maclaurin and Taylor series([6]) are not widely used of their polynomial behavior.
Formulas of derivative's series expansion of Indefinite and Definite Integral, which were concluded from this work, allow to find approximate solution in symbolic math with controlled accuracy.

This could be an useful tool for integration multi-parametric functions and finding integrals in Differential Equations.

2. Definitions

Definition 1. *Let $i \in \{0, \mathbb{N}\}$. We say that i-**order derivative** of $f(x)$ function is defined as*

$$f^{(i)} \equiv \frac{d^i f}{dx^i} \tag{1}$$

or recursive

$$\begin{cases} f^{(0)} = f \\ f^{(i)} = \frac{df^{(i-1)}}{dx}, i \in \mathbb{N} \end{cases} \tag{2}$$

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$$F^{(i)} \equiv \underbrace{\int \cdots \int}_{i} f(dx)^i \tag{3}$$

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$$\begin{cases} F^{(0)} = f \\ F^{(i)} = \int F^{(i-1)} dx, i \in \mathbb{N} \end{cases} \tag{4}$$

3. Lemma of Integration of complex function

Lemma 1. *Let u, v are functions. If $\forall i \in \{0, \mathbb{N}\} : \exists u^{(i)}, \exists V^{(i)}$: exists all i-order derivatives of $\boldsymbol{u}$ function and exists all i-order integrals of $\boldsymbol{v}$ function then*

$$\int u dv = \sum_{i=0}^{\infty} (-1)^i \, u^{(i)} V^{(i)} \tag{5}$$

Proof. Knowing ([1]) that

$$d \int f(x) \, dx = f(x) \, dx$$

and

$$\int df = \int df \frac{dx}{dx} = \int \frac{df}{dx} dx$$

and using (1) and (3), it's easy to show that following equality of integral $\int vdu$ is

$$\int vdu = \int \frac{du}{dx} d \left(\int vdx \right) \Longleftrightarrow \int V^{(0)} du^{(0)} = \int u^{(1)} dV^{(1)} \tag{6}$$

Furthermore, integral with i-order derivative and i-order integra is

$$\int V^{(i)} du^{(i)} = \int \frac{du^{(i)}}{dx} d \left(\int V^{(i)} dx \right) = \int u^{(i+1)} dV^{(i+1)} \tag{7}$$

Using ([1])

$$\int udv = uv - \int vdu \tag{8}$$

and substituting (6) into it, we get

$$\int u^{(0)} dV^{(0)} = u^{(0)} V^{(0)} - \int u^{(1)} dV^{(1)} \tag{9}$$

and generalized case, using (7):

$$\int u^{(i)} dV^{(i)} = u^{(i)} V^{(i)} - \int u^{(i+1)} dV^{(i+1)} \tag{10}$$

Let's evaluate the integral $\int udv$ with (9) and (10) using mathematical induction:

$$\int udv = u^{(0)} V^{(0)} - \int u^{(1)} dV^{(1)} = u^{(0)} V^{(0)} - u^{(1)} V^{(1)} + \int u^{(2)} dV^{(2)} = \dots$$
$$= u^{(0)} V^{(0)} - u^{(1)} V^{(1)} + u^{(2)} V^{(2)} - u^{(3)} V^{(3)} + u^{(4)} V^{(4)} - u^{(5)} V^{(5)} + \dots \tag{11}$$

Thus,

$$\int udv = \sum_{i=0}^{\infty} (-1)^i \, u^{(i)} V^{(i)} \tag{12}$$

□

Forms. The formula (5) could be written in different terms. Standard form:

$$\int u dv = \sum_{i=0}^{\infty} (-1)^i \frac{d^i u}{dx^i} \underbrace{\int \cdots \int}_{i} v(dx)^i \tag{13}$$

Form with partial expantion:

$$\int u dv = \sum_{i=0}^{n-1} (-1)^i u^{(i)} V^{(i)} + (-1)^n \int u^{(n)} dV^{(n)} \tag{14}$$

Example 1. Let's find integral of $\int \frac{e^x}{x} dx$:
By the Lemma 1 with (5)

$$\int \frac{e^x}{x} dx = \left| \begin{matrix} v = e^x, & V^{(i)} = e^x, C_i = 0, C_\infty = C \\ u = \frac{1}{x}, & u^{(i)} = (-1)^i \, i! \frac{1}{x^{i+1}} \end{matrix} \right| = e^x \sum_{i=0}^{\infty} \frac{i!}{x^{i+1}} + C \tag{15}$$

Example 2. Let's find integral of $\int \frac{e^x}{x^n} dx$:
By the Lemma 1 with (5)

$$\begin{aligned} \int \frac{e^x}{x^n} dx = \\ & \left| \begin{matrix} v = e^x, & V^{(i)} = e^x, C_i = 0, C_\infty = C \\ u = \frac{1}{x^n}, & u^{(i)} = (-1)^i \frac{(n-1+i)!}{(n-1)!} \frac{1}{x^{i+n}} \end{matrix} \right| = \\ & \frac{e^x}{(n-1)!} \sum_{i=0}^{\infty} \frac{(n-1+i)!}{x^{i+n}} + C \end{aligned} \tag{16}$$

4. Formulas of Integration

Theorem 1 (Main)**.** *Let $f(x)$ is function. If $\forall i \in \{0, \mathbb{N}\} : \exists f^{(i)}(x)$ exists all i-order derivatives of this function, then*

$$\int f(x) dx = \sum_{i=0}^{\infty} (-1)^i f^{(i)}(x) \frac{x^{i+1}}{(i+1)!} + C \tag{17}$$

Proof. We can easily find integral by Lemma 1 and (5)

$$\begin{aligned} \int f(x) dx = F^{(1)}(x) = \left| \begin{matrix} v = x & V^{(i)} = \frac{x^{i+1}}{(i+1)!}, C_i = 0, C_\infty = C \\ u = f & u^{(i)} = f^{(i)} \end{matrix} \right| = \\ \sum_{i=0}^{\infty} (-1)^i f^{(i)}(x) \frac{x^{i+1}}{(i+1)!} + C \end{aligned} \tag{18}$$

$\square$

Forms. The formula (17) could be written in different terms. Standard form:

$$\int f(x)dx = f(x) + \sum_{i=1}^{\infty} (-1)^i \frac{d^i f(x)}{dx^i} \frac{x^{i+1}}{(i+1)!} + C \tag{19}$$

Checking. Let's check, if $(\int f(x)dx)' = f(x)$:
From (17) :

$$\left(\sum_{i=0}^{\infty} (-1)^i f^{(i)} \frac{x^{i+1}}{(i+1)!} + C \right)' = \tag{20}$$
$$= |(fg)' = f'g + fg'| =$$
$$= \sum_{i=0}^{\infty} (-1)^i f^{(i)} \frac{(i+1)x^i}{(i+1)!} + \sum_{i=0}^{\infty} (-1)^i f^{(i+1)} \frac{x^{i+1}}{(i+1)!}$$
$$= f(x) + \sum_{i=1}^{\infty} (-1)^i f^{(i)} \frac{x^i}{i!} - \sum_{i=1}^{\infty} (-1)^i f^{(i)} \frac{x^i}{i!}$$
$$= f(x) \tag{21}$$

Example 3. Let's find integral of $\int xdx$:
By the Theorem 1 with (17)

$$\int xdx = \left| \begin{array}{c} f^{(0)} = x \\ f^{(1)} = 1 \\ f^{(i)} = 0, i > 1, i \in \mathbb{N} \end{array} \right| = xx - \frac{x^2}{2} + C = \frac{x^2}{2} + C \tag{22}$$

Example 4. Let's find integral of $\int e^x dx$:
By the Theorem 1 with (17)

$$\int e^x dx = \left| f^{(i)} = e^x, i \in \mathbb{N} \right| = e^x \sum_{i=0}^{\infty} (-1)^i \frac{x^{i+1}}{(i+1)!} + C \tag{23}$$

Using Taylor series ([1])

$$\sum_{i=1}^{\infty} (-1)^{i+1} \frac{x^i}{i!} = e^{-x}(e^x - 1) \tag{24}$$

integral (23) could be rewritten in

$$\int e^x dx = e^x e^{-x}(e^x - 1) + C = e^x + C' \tag{25}$$

Theorem 2. *Let $f(x)$ is function. If $\forall i \in \{0, \mathbb{N}\} : \exists f^{(i)}(x)$ exists all i-order derivatives of this function, then*

$$\int_a^b f(x)dx = \sum_{i=0}^{\infty} (-1)^i \frac{b^{i+1} f^{(i)}(b) - a^{i+1} f^{(i)}(a)}{(i+1)!} \tag{26}$$

Proof. By First Fundamental Theorem of Calculus [1] and by Teorem 1 with (17) we get

$$\begin{aligned}
&\int_a^b f(x)dx = \\
&\quad = \left| \int_a^b f(x)dx = F^{(1)}(b) - F^{(1)}(a) \right| \\
&\quad = \sum_{i=0}^{\infty} (-1)^i f^{(i)}(b) \frac{b^{i+1}}{(i+1)!} + C - \left(\sum_{i=0}^{\infty} (-1)^i f^{(i)}(a) \frac{a^{i+1}}{(i+1)!} + C \right) \\
&\quad = \sum_{i=0}^{\infty} (-1)^i \frac{b^{i+1} f^{(i)}(b) - a^{i+1} f^{(i)}(a)}{(i+1)!} \qquad (27)
\end{aligned}$$

□

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